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Stochastic averaging of MDOF quasi integrable Hamiltonian systems under wide-band random excitation

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Abstract

A stochastic averaging method for predicting the response of multi-degree-of-freedom (MDOF) quasi-integrable Hamiltonian systems to external and/or parametric wide-band random excitations is proposed. The motion equations governing a MDOF quasi-integrable Hamiltonian system is reduced to a set of averaged Itô stochastic differential equations via stochastic averaging and the associated averaged Fokker–Planck–Kolmogorov (FPK) equation is derived. The joint probability density of amplitudes and/or energies is obtained from solving the FPK equation. One example is given to illustrate the proposed method in detail and the effectiveness of the proposed method is verified via comparing the analytical results with those from Monte Carlo simulation.

1. Introduction

Strongly nonlinear systems subject to random excitation occur very often in science and engineering. A versatile and powerful approximate approach to predicting the response of strongly nonlinear systems to random excitation is the stochastic averaging method. The stochastic averaging method of energy envelope was first proposed by Landa and Stratonovich [1] for stochastically excited Duffing oscillator and by Khasminskii [2] for two-dimensional quasi-Hamiltonian systems, respectively. A more rigorous formulation for this method was developed by Zhu [3] and Zhu and Lin [4] based on a theorem due to Khasminskii [5] and applied to generally quasi-conservative nonlinear oscillators of single-degree-of-freedom (SDOF) subject to external and (or) parametric excitations of correlated Gaussian white noises. The stochastic averaging method for SDOF strongly nonlinear oscillators subject to external and/or parametric excitations of wide-band random processes was also developed [6–9]. The method has been applied to predict the response of Duffing–van der Pol oscillator under both external and parametric excitations of wide-band stationary random processes.

The state space of physical or engineering dynamical systems is generally more than two-dimensional. However, the stochastic averaging method of energy envelope was developed only for SDOF nonlinear

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systems. In the last decade, the stochastic averaging method for quasi-Hamiltonian systems has been proposed by the present second author and his co-workers [10–12]. It is applicable to MDOF strongly nonlinear oscillators with light dampings subject to weak external and/or parametric Gaussian white noise excitations. The dimension and form of the averaged Itô stochastic differential equations depend on the integrability and resonance of the associated Hamiltonian systems. The stochastic averaging method for quasi-integrable Hamiltonian systems has been applied to study the first-passage failure [13] and optimal bounded control [14]. By applying the stochastic averaging method for quasi-integrable Hamiltonian systems to studying the motion of the active Brownian particles, an analytical stationary solution has been obtained [15,16].

In the present paper, the stochastic averaging method for MDOF quasi-integrable Hamiltonian systems is extended to the case of external and parametric excitations of wide-band random processes. One example is given to illustrate the proposed method in detail and the effectiveness of the method is verified by comparing the results obtained from analysis with those from digital simulation.

2. Stochastic averaging method

A vibratory system usually consists of conservative oscillator, damping and excitation. A general class of n-DOF strongly nonlinear non-gyroscopic vibratory system under random excitation can be described by the following Lagrange equations:

$$\ddot{X}_i + c_{ij}(\mathbf{X},\dot{\mathbf{X}})\dot{X}_j + \frac{\partial U(X)}{\partial X_i} = f_{ik}(\mathbf{X},\dot{\mathbf{X}})\xi_k(t),$$
$$i,j = 1,2,\ldots,n, \quad k = 1,2,\ldots,m, \tag{1}$$

where $\mathbf{X} = [X_1, X_2, \ldots, X_n]^{\mathrm{T}}$ are generalized displacements; $\dot{\mathbf{X}} = [\dot{X}_1, \dot{X}_2, \ldots, \dot{X}_n]^{\mathrm{T}}$ are generalized velocities; $c_{ij}(\mathbf{X},\dot{\mathbf{X}})$ denote the coefficients of quasi-linear dampings; $f_{ij}(\mathbf{X},\dot{\mathbf{X}})$ denote amplitudes of random excitations; $U(\mathbf{X})$ is the potential function of the oscillator; $\xi_k(t)$ are stationary wide-band random processes with correlation functions $R_{kl}(\tau) = E[\xi_k(t)\xi_l(t+\tau)]$ or spectral densities $S_{kl}(\omega)$. In most cases, dampings are light and random excitations are weak. Letting $Q_i = X_i$, $P_i = \dot{X}_i$, the Lagrange equations in Eq. (1) can be converted into the following equations for quasi-Hamiltonian system:

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i} - \varepsilon c_{ij}\frac{\partial H}{\partial P_j} + \varepsilon^{1/2} f_{ik}\xi_k(t),$$
$$i,j = 1,2,\ldots,n, \quad k = 1,\ldots,2,m, \tag{2}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H = H(\mathbf{Q},\mathbf{P}) = \sum_{i=1}^{n} P_i^2/2 + U(\mathbf{Q})$ is twice differentiable Hamiltonian representing the total energy of the system; ε is a small parameter; $-\varepsilon c_{ij} = -\varepsilon c_{ij}(\mathbf{Q},\mathbf{P})$ are the coefficients of lightly quasi-linear dampings; $\varepsilon^{1/2} f_{ik} = \varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})$ are the amplitudes of weak random external and/or parametric excitations.

When $\varepsilon = 0$, system (2) is reduced to a Hamiltonian system which can be integrable or non-integrable. A Hamiltonian system of n DOF is said to be integrable or completely integrable if there exist n independent motion integrals, H_1, $H_2,\ldots,H_n$, which are in involution. If the Hamiltonian H in Eq. (2) is separable, i.e.,

$$H(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} H_i(q_i, p_i), \tag{3}$$

and H_i are of the form:

$$H_i(q_i, p_i) = p_i^2/2 + U_i(q_i), \tag{4}$$

where

$$U_i(q_i) = \int_0^{q_i} g(u)\,\mathrm{d}u, \tag{5}$$

then the system governed by Eq. (2) is a quasi-integrable Hamiltonian system.

Furthermore, if functions $g_i(q_i)$ and $U_i(q_i)$ satisfy the following four conditions:

(i) $g_i(0) = 0$,
(ii) there exists a point $q_{10} > 0$ such that $g_i(q) \neq 0$ and $U_i(q_{i0}) > 0$,
(iii) a point $q_{i1} < 0$ can be found such that $g(q_{i1}) \neq 0$ and $U_i(q_{i1}) = U_i(q_{i0})$,
(iv) for all $q_{i1} < q_i < q_{i0}$, $U_i(q_i) < U_i(q_{i0})$,

then the Hamiltonian system associated with Eq. (2) has the following periodic solution [17]:

$$q_i(t) = a_i \cos \varphi_i(t) + b_i, \quad \dot{q}_i(t) = -a_i v_i(a_i, \varphi_i) \sin \varphi_i(t), \quad \varphi_i(t) = \psi_i(t) + \theta_i, \tag{6}$$

where θ_i is the initial phase angle of the ith degree of freedom; $v_i(a_i, \varphi_i) = \mathrm{d}\varphi_i(t)/\mathrm{d}t$ is the instantaneous angle frequency; a_i are the amplitudes and b_i is the symmetric center coordinate of q_i. $\cos \varphi_i(t)$ and $\sin \varphi_i(t)$ are called generalized harmonic functions.

From Eqs. (5) and (6) one can obtain the following equations:

$$U_i(a_i + b_i) = U_i(-a_i + b_i), \tag{7}$$

$$v_i(a_i, \varphi_i) = \frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \frac{\mathrm{d}\psi_i}{\mathrm{d}t} = \sqrt{\frac{2[U_i(a_i + b_i) - U_i(a_i \cos \varphi_i + b_i)]}{a_i^2 \sin^2 \varphi_i}}, \tag{8}$$

a_i, b_i and $v_i(a_i, \varphi_i)$ can be obtained by directly or indirectly solving Eqs. (7) and (8). It is noted that the symmetric center coordinate b_i can be expressed as function of amplitude a_i and the instantaneous angle frequency. $v_i(a_i, \varphi_i)$ is a function of a_i and phase angles φ_i. In the following, a new variable h_i is introduced to denote $\mathrm{d}b_i/\mathrm{d}a_i$. The explicit expression for h_i can be obtained from Eq. (7) as follows:

$$h_i = \frac{\mathrm{d}b_i}{\mathrm{d}a_i} = \frac{g(-a_i + b_i) + g(a_i + b_i)}{g(-a_i + b_i) - g(a_i + b_i)}. \tag{9}$$

The instantaneous angle frequency $v_i(a_i, \varphi_i)$ can be expanded into Fourier series with respect to φ_i as

$$v_i(a_i, \varphi_i) = \tfrac{1}{2} c_{i,0}(a_i) + \sum_{j=1}^{\infty} c_{i,j}(a_i) \cos j\varphi_i, \tag{10}$$

where $c_{i,0}(a_i)/2$ is the mean angle frequency $\bar{\omega}_i(a_i)$ of the ith degree of freedom, i.e.,

$$\bar{\omega}_i(a_i) = \frac{1}{2} c_{i,0}(a_i) = \frac{1}{2\pi} \int_0^{2\pi} v_i(a_i, \varphi_i)\, \mathrm{d}\varphi_i. \tag{11}$$

The following approximate relations will be used in averaging operation:

$$\psi_i(t) \approx \bar{\omega}_i(a)t, \quad \varphi_i(t) \approx \bar{\omega}_i(a_i)t + \theta_i. \tag{12}$$

When ε is small, the response of system (2) is random periodic and of the form:

$$Q_i(t) = A_i \cos \Phi_i(t) + B_i, \quad \dot{Q}_i(t) = -A_i V_i(A_i, \Phi_i) \sin \Phi_i(t), \quad \Phi_i(t) = \Psi_i(t) + \Theta_i(t), \tag{13}$$

where

$$V_i(A_i, \Phi_i) = \frac{\sqrt{2[U_i(A_i + B_i) - U_i(A_i \cos \Phi_i + B_i)]}}{|A_i \sin \Phi_i|}. \tag{14}$$

Regarding Eq. (13) as a transformation from Q_i, $\dot{Q}_i$ to A_i, Φ_i, one can obtain the following equations for $A_i(t)$ and $\Phi_i(t)$:

$$\dot{A}_i = \varepsilon F_i^A(\mathbf{A}, \mathbf{\Phi}) + \varepsilon^{1/2} G_{ik}^A(\mathbf{A}, \mathbf{\Phi}) \xi_k(t), \quad \dot{\Phi}_i = \varepsilon F_i^{\Phi}(\mathbf{A}, \mathbf{\Phi}) + \varepsilon^{1/2} G_{ik}^{\Phi}(\mathbf{A}, \mathbf{\Phi}) \xi_k(t),$$
$$i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m, \tag{15}$$

where

$$\mathbf{A} = [A_1, A_2, \ldots, A_n]^{\mathrm{T}}, \quad \mathbf{\Phi} = [\Phi_1, \Phi_2, \ldots, \Phi_n]^{\mathrm{T}},$$
$$F_i^A = \frac{A_i}{g_i(A_i + B_i)(1 + h_i)} \left[c_{ij}(\mathbf{A}, \mathbf{\Phi}) A_j V_j(A_j, \Phi_j) \sin \Phi_j\right] v_i(A_i, \Phi_i) \sin \Phi_i,$$
$$F_i^{\Phi} = \frac{1}{g_i(A_i + B_i)(1 + h_i)} \left[c_{ij}(\mathbf{A}, \mathbf{\Phi}) A_j V_j(A_j, \Phi_j) \sin \Phi_j\right] v_i(A_i, \Phi_i)(\cos \Phi_i + h_i),$$
$$G_i^A = \frac{A_i}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(\mathbf{A}, \mathbf{\Phi}) v_i(A_i, \Phi_i) \sin \Phi_i,$$
$$G_i^{\Phi} = \frac{1}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(\mathbf{A}, \mathbf{\Phi}) v_i(A_i, \Phi_i)(\cos \Phi_i + h_i). \tag{16}$$

By substituting A_i and B_i into Eq. (9) to replace a_i and b_i, h_i in Eq. (16) can be obtained as follows:

$$h_i = \frac{g(-A_i + B_i) + g(A_i + B_i)}{g(-A_i + B_i) - g(A_i + B_i)}. \tag{17}$$

Based on the Stratonovich–Khasminskii limit theorem [1,18], the amplitude vector process $\mathbf{A}(t)$ in Eq. (15) converges weakly to an n-dimensional vector diffusion process as $\varepsilon \to 0$ in a time internal [0, T], where $T \sim 0(\varepsilon^{-1})$. This limiting diffusion process is governed by the following averaged Itô equations:

$$\mathrm{d}A_i = m_i(\mathbf{A})\,\mathrm{d}t + \sigma_{ik}(\mathbf{A})\,\mathrm{d}B_k(t), \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m, \tag{18}$$

where the drift and diffusion coefficients are:

$$m_i(\mathbf{A}) = \varepsilon \left\langle F_i^A + \int_{-\infty}^{0} \left(\left(\partial G_{ik}^A / \partial A_j\right)\Big|_t G_{jl}^A\Big|_{t+\tau} + \left(G_{ik}^A / \Phi_j\right)\Big|_t G_{jl}^{\Phi}\Big|_{t+\tau} \right) R_{kl}(\tau)\,\mathrm{d}\tau \right\rangle_t,$$
$$b_{ij}(\mathbf{A}) = \sigma_{ik}(\mathbf{A})\sigma_{jk}(\mathbf{A}) = \varepsilon \left\langle \int_{-\infty}^{\infty} \left(G_{ik}^A\Big|_t G_{jl}^A\Big|_{t+\tau} \right) R_{kl}(\tau)\,\mathrm{d}\tau \right\rangle_t \tag{19}$$

in which $\langle \cdot \rangle_t$ denotes the time-averaging operation, i. e.,

$$< \cdot >_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T < \cdot > \mathrm{d}t. \tag{20}$$

To obtain the explicit expressions for m_i and b_{ij}, first expand F_i^A, ${F_{ik}}^A$, ${G_{ik}}^{\Phi}$ into n-fold Fourier series with respect to Φ_i, integrate with respect to τ and then average with respect to Φ_i using Eq. (12). The FPK equation governing the transition probability density of $\mathbf{A}(t)$ can be derived from Itô Eq. (18) as follows:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A_i}[m_i(\mathbf{a})p] + \frac{1}{2}\frac{\partial^2}{\partial A_i \partial A_j}\left[b_{ij}(\mathbf{a})p\right], \quad i,j = 1, 2, \ldots, n, \tag{21}$$

where $p = p(\mathbf{a}, t|\mathbf{a}_0)$ is the transition probability density of amplitudes $\mathbf{A}(t)$ with initial condition:

$$p(\mathbf{a}, 0|\mathbf{a}_0) = \delta(\mathbf{a} - \mathbf{a}_0), \tag{22}$$

or $p = p(\mathbf{a}, t)$ is the probability density of $\mathbf{A}(t)$ with initial condition:

$$p(\mathbf{a}, 0) = p(\mathbf{a}_0). \tag{23}$$

In some cases the averaged Itô equations for the first integrals H_i (energies of various degrees of freedom) in Eq. (4) are preferred. They can be obtained from Eq. (18) by using Itô differential rule and relations:

$$H_i = U_i(A_i + B_i). \tag{24}$$

The result is

$$\mathrm{d}H_i = \bar{m}_i(\mathbf{H})\,\mathrm{d}t + \bar{\sigma}_{ik}(\mathbf{H})\,\mathrm{d}B_k(t), \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m, \tag{25}$$

where $\mathbf{H} = [H_1, H_2, \ldots, H_n]^{\mathrm{T}}$ is the energy vector and

$$\bar{m}_i(\mathbf{H}) = \left[g_i(A_i+B_i)(1+h_i)m_i(\mathbf{A}) + \frac{1}{2}\frac{\mathrm{d}}{\partial A_i}[g_i(A_i+B_i)(1+h_i)]\sigma_{ik}(\mathbf{A})\sigma_{ik}(\mathbf{A}) \right]\bigg|_{A_i=U_i^{-1}(H_i)-B_i},$$

$$\bar{b}_{ij}(\mathbf{H}) = \bar{\sigma}_{ik}(\mathbf{H})\bar{\sigma}_{jk}(\mathbf{H}) = \left[g_i(A_i+B_i)g_j(A_j+B_j)(1+h_i)(1+h_j)\sigma_{ik}(\mathbf{A})\sigma_{jk}(\mathbf{A}) \right]\big|_{A_i=U_i^{-1}(H_i)-B_i}. \tag{26}$$

The FPK equation associated with Itô Eq. (25) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial H_r}[\bar{m}_r(\mathbf{H})p] + \frac{1}{2}\frac{\partial^2}{\partial H_r \partial H_s}\left[\bar{b}_{rs}(\mathbf{H})p\right], \quad r,s = 1,2,\ldots,n, \tag{27}$$

where $p = p(\mathbf{H}, t|\mathbf{H}_0)$ is the transition probability density of energy vector process $\mathbf{H}(t)$ with initial condition:

$$p(\mathbf{H}, 0|\mathbf{H}_0) = \delta(\mathbf{H} - \mathbf{H}_0), \tag{28}$$

or $p = p(\mathbf{H}, t)$ is the probability density of energy vector process $\mathbf{H}(t)$ with the initial condition:

$$p(\mathbf{H}, 0) = p(\mathbf{H}_0), \tag{29}$$

depending on whether an initial state or an initial probability density is specified. Letting the left-hand side of FPK Eq. (27) vanish, i.e., $\partial p/\partial t = 0$, one obtains the stationary (reduced) FPK equation. The solution for it usually can be solved only numerically if the coefficients (26) are very complicated.

For quasi-integrable Hamiltonian system with Hamiltonian of form (3), a derivation similar to that of single degree-of-freedom quasi-Hamiltonian systems [1,3] leads to the following stationary probability density for system state:

$$p(\mathbf{q}, \mathbf{p}) = \left(p(\mathbf{H})/T(\mathbf{H})\right)\big|_{H_r=H_r(q,p)}. \tag{30}$$

where $T(\mathbf{H})$ is the multiplier of the n periods for n oscillators.

The statistics of the stationary response of system (2) such as the marginal stationary probability density $p(q_r)$ and the mean-square value $E[Q_r^2]$ can then be obtained from Eq. (30).

Note that $\mathbf{A}(t)$ in Eq. (18) and $\mathbf{H}(t)$ in Eq. (25) are homogeneous diffusion processes. So, the diffusion processes theory can be applied to them. Furthermore, the dimension of averaged equations in Eq. (18) or (25) is only a half of that of Eq. (2) and in the averaged equations only the slowly varying processes $A_i(t)$ or $H_i(t)$are retained. Note that the time averaging in Eq. (19) smoothens out only the rapidly temporal fluctuation in the response but not the transient response. So, the averaged Eqs. (18) and (25) can depict the averaged transient response as well as averaged stationary response.

3. Example of application

As an application example of the proposed stochastic averaging method, consider two Duffing oscillators coupled by both linear dampings and external and parametric excitations of stationary wide-band random processes. The equations of the system are of the form:

$$\ddot{X}_1 + \beta_{11}\dot{X}_1 + \beta_{12}\dot{X}_2 + \omega_1^2 X_1 + \alpha_1 X_1^3 = X_1\xi_{11}(t) + \xi_{12}(t),$$

$$\ddot{X}_2 + \beta_{21}\dot{X}_1 + \beta_{22}\dot{X}_2 + \omega_2^2 X_2 + \alpha_2 X_2^3 = X_2\xi_{21}(t) + \xi_{22}(t), \tag{31}$$

where β_{ij}, ω_i, α_i, $(i = 1, 2)$ are constants; $\xi_{i1}(t)$ and $\xi_{i2}(t)$are independent stationary random processes with auto spectral densities:

$$S_{ij}(\omega) = \frac{D_{ij}}{\pi(\omega^2 + \Omega_{ij}^2)}, \quad i,j = 1,2, \tag{32}$$

D_{ij} and Ω_{ij} are constants. β_{ij}, D_{ij} are assumed of the same order of small parameter ε. A single Brownian particle under random perturbation moving in the following potential landscape (see Fig. 1):

$$U(X_1, X_2) = \tfrac{1}{2}\omega_1^2 X_1^2 + \tfrac{1}{4}\alpha_1 X_1^4 + \tfrac{1}{2}\omega_2^2 X_2^2 + \tfrac{1}{4}\alpha_2 X_2^4 \tag{33}$$

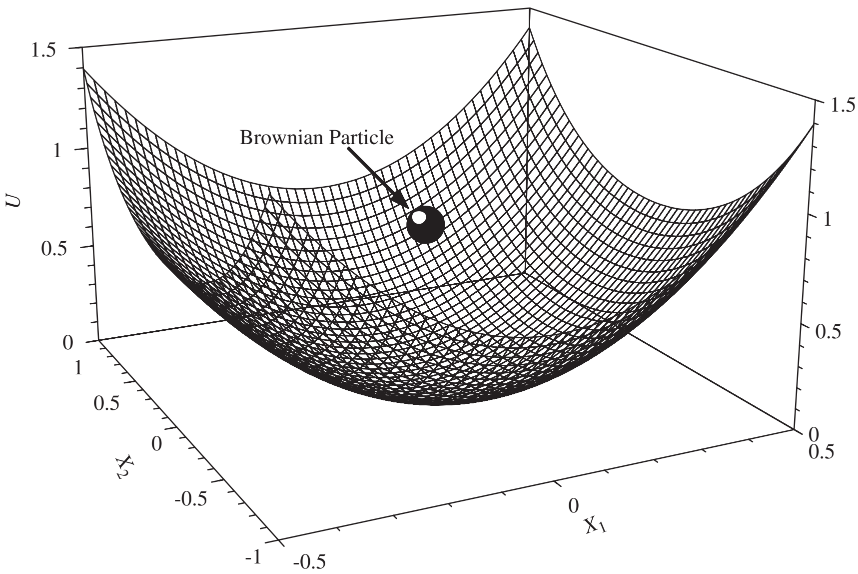

Fig. 1. A Brownian particle moving in two-dimensional potential landscape governed by Eq. (33).

is a physical model of Eq. (31). Eq. (31) can also describe the motion of the first two modes of a thin flexible plate under normal and axial random loadings.

Letting $X_1 = Q_1$, $\dot{X}_1 = P_1$, $X_2 = Q_2$, $\dot{X}_2 = P_2$, Eq. (31) can be converted into the form of Eq. (2). The Hamiltonian associated with system (31) is

$$H = H_1 + H_2, \quad H_i = \tfrac{1}{2}p_i^2 + \tfrac{1}{2}\omega_i^2 q_i^2 + \tfrac{1}{4}\alpha_i q_i^4, \quad i = 1, 2, \tag{34}$$

which are of the form of Eqs. (3) and (4) with

$$U_i(q_i) = \tfrac{1}{2}\omega_i^2 q_i^2 + \tfrac{1}{4}\alpha_i q_i^4. \tag{35}$$

Since $U_i(q_i)$ is symmetrical with respect to $q_i = 0$, B_i and h_i in Eq. (17) vanish, i.e.,

$$g_i = g_i(A_i) = \omega_i^2 A_i + \alpha_i A_i^3, \quad B_i = h_i = 0. \tag{36}$$

The sub-Hamiltonian systems associated with system (31) have periodic solutions of the form of Eq. (6) in whole phase plane (q_i, p_i) if $\omega_i^2, \alpha_i > 0$. The instantaneous frequencies are:

$$\begin{aligned} v_i(A_i, \varphi_i) &= [(\omega_i^2 + 3\alpha_i A_i^2/4)(1 + \lambda_i \cos 2\varphi_i)]^{1/2}, \\ \lambda_i &= (\alpha_i A_i^2/4)/(\omega_i^2 + 3\alpha_i A_i^2/4) \leqslant 1/3, \end{aligned} \tag{37}$$

v_i can be expand into Fourier series. To simplify the calculation, the series are truncated and v_i are approximated by the following finite sums with a relative error less than 0.03%:

$$v_i(A_i, \varphi_i) \approx c_{i,0}(A_i)/2 + c_{i,2}(A_i)\cos 2\varphi_i + c_{i,4}(A_i)\cos 4\varphi_i + c_{i,6}(A_i)\cos 6\varphi_i, \tag{38}$$

where

$$\begin{aligned} c_{i,0} &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(1 - \lambda_i^2/16), \quad c_{i,2} = (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(\lambda_i/2 + 3\lambda_i^3/64), \\ c_{i,4} &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(-\lambda_i^2/16), \quad b_{i,6} = (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(\lambda_i^3/64). \end{aligned} \tag{39}$$

The averaged frequencies are:

$$\bar{\omega}_i(A_i) = c_{i,0}(A_i)/2. \tag{40}$$

By using transformations (13), the following equations for A_i and Φ_i are obtained from Eq. (31):

$$\dot{A}_i = F_i^A(\mathbf{A},\mathbf{\Phi}) + G_{i1}^A(\mathbf{A},\mathbf{\Phi})\xi_{i1}(t) + G_{i2}^A(\mathbf{A},\mathbf{\Phi})\xi_{i2}(t),$$
$$\dot{\Phi}_i = F_i^{\Phi}(\mathbf{A},\mathbf{\Phi}) + G_{i1}^{\Phi}(\mathbf{A},\mathbf{\Phi})\xi_{i1}(t) + G_{i2}^{\Phi}(\mathbf{A},\mathbf{\Phi})\xi_{i2}(t), \tag{41}$$

where

$$F_i^A = -A_i\beta_{ij}A_jv_j \sin\Phi_j v_i \sin\Phi_j/g_i, \quad G_{i1}^A = -A_i^2 v_i \sin\Phi_i \cos\Phi_i/g_i,$$
$$G_{i2}^A = -A_i v_i \sin\Phi_i/g_i, \quad G_{i1}^{\Phi} = -A_i v_i \cos^2\Phi_i/g_i, \quad G_{i2}^{\Phi} = -v_i \cos\Phi_i/g_i. \tag{42}$$

By inserting Eq. (38) into Eq. (42) and completing the stochastic averaging, the following averaged Itô equations are obtained from Eq. (41):

$$\mathrm{d}A_i = m_i(\mathbf{A})\,\mathrm{d}t + \sigma_{i1}(\mathbf{A})\,\mathrm{d}B_{i1}(t) + \sigma_{i2}(\mathbf{A})\,\mathrm{d}B_{i2}(t), \tag{43}$$

where the drift and diffusion coefficients are obtained by using formulas in Eq. (19) as follows:

$$m_i(A) = \frac{-A_i^2}{8g_i}\beta_{ii}\left(4\omega_i^2 + \frac{5}{2}\alpha_i A_i^2\right) + \frac{\pi A_i}{32g_i}$$
$$\times \sum_{n=2}^{\infty}\left\{A_i\left(c_{i,n-2} - c_{i,n+2}\right)\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left(\frac{A_i^2\left(c_{i,n-2} - c_{i,n+2}\right)}{g_i}\right) + \frac{nA_i}{g_i}\left(c_{i,n-2} + 2c_{i,n} + c_{i,n+2}\right)\right]S_{i1}(n\bar{\omega}_i)\right.$$
$$\left. + 4\left(c_{i,n-2} - c_{i,n}\right)\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left(\frac{A_i\left(c_{i,n-2} - c_{i,n}\right)}{g_i}\right) + \frac{n}{g_i}\left(c_{i,n-2} + c_{i,n}\right)\right]S_{i2}((n-1)\bar{\omega}_i)\right\},$$
$$b_{ii}(\mathbf{A}) = \frac{\pi A_i^2}{16g_i^2}\sum_{n=2}^{\infty}\left\{A_i^2\left(c_{i,n-2} - c_{i,n+2}\right)^2 S_{i1}(n\bar{\omega}_i) + 4\left(c_{i,n-2} - c_{i,n}\right)^2 S_{i2}((n-1)\bar{\omega}_i)\right\},$$
$$b_{12}(\mathbf{A}) = b_{21}(A) = 0, \quad i = 1, 2, \quad n = 2, 4, 6, \ldots. \tag{44}$$

The relations between A_i and H_i obtained from Eq. (34) are:

$$A_i = U_i^{-1}(H_i) = \sqrt{\left(\sqrt{\omega_i^4 + 4\alpha_i H_i} - \omega_i^2\right)\Big/\alpha_i}. \tag{45}$$

The averaged Itô equations for H_i are then:

$$\mathrm{d}H_i = \bar{m}_i(\mathbf{H})\,\mathrm{d}t + \bar{\sigma}_{i1}(\mathbf{H})\,\mathrm{d}B_{i1}(t) + \bar{\sigma}_{i2}(\mathbf{H})\,\mathrm{d}B_{i2}(t), \tag{46}$$

where

$$\bar{m}_i(\mathbf{H}) = \left[\left(\omega_i^2 A_i + \alpha_i A_i^3\right)m_{i1}(\mathbf{A}) + \frac{1}{2}\left(\omega_i^2 + 3\alpha_i A_i^2\right)b_{ii}^2(\mathbf{A})\right]\Big|_{A_i = U_i^{-1}(H_i)},$$
$$\bar{b}_{ii}(\mathbf{H}) = \left[\left(\omega_i^2 A_i + \alpha_i A_i^3\right)^2 b_{ii}^2(\mathbf{A})\right]\Big|_{A_i = U_i^{-1}(H_i)},$$
$$\bar{b}_{12}(\mathbf{H}) = \bar{b}_{21}(\mathbf{H}) = 0. \tag{47}$$

Following the discussion of the last section, the stationary probability density of sub-Hamiltonians $H_1(t)$, $H_2(t)$ of system (31) can be obtained from solving the following stationary FPK equation:

$$-\frac{\partial}{\partial H_i}[\bar{m}_i(\mathbf{H})p] + \frac{1}{2}\frac{\partial^2}{\partial H_i^2}\left[\bar{b}_{ii}(\mathbf{H})p\right] = 0. \tag{48}$$

The associated boundary condition is

$$p(H_1, H_2 \to \infty) = p(H_1 \to \infty, H_2) = 0. \tag{49}$$

Although H_i may vary in infinite interval $[0, \infty)$, the following finite rectangle domain is used in numerically solving the stationary FPK equation (Eq. (48)):

$$0 \leqslant H_1 \leqslant H_{c1}, \quad 0 \leqslant H_2 \leqslant H_{c2}, \tag{50}$$

where H_{c1} and H_{c2} are large enough so that

$$p(H_1 = H_{c1}, H_2) \approx 0, \quad p(H_1, H_2 = H_{c2}) \approx 0, \quad \int_{H_{c1}}^{\infty} \int_{H_{c2}}^{\infty} p(H_1, H_2)\,\mathrm{d}H_2\,\mathrm{d}H_1 \approx 0. \tag{51}$$

The solution of FPK equation (Eq. (48)) is also subject to the following normalization condition:

$$\int_0^{H_{c1}} \int_0^{H_{c2}} p(H_1, H_2)\,\mathrm{d}H_2\,\mathrm{d}H_1 \approx 1. \tag{52}$$

The stationary FPK equation (Eq. (48)) is an elliptic partial differential equation and can be numerically solved together with the boundary condition (51) and normalization condition (52) by using Peaceman–Rachford scheme of finite difference method. By using Eq. (30) and noting that the $T(H_1, H_2)$ is essentially the multiplier of the two periods $2\pi/\bar{\omega}_1$ and $2\pi/\bar{\omega}_2$ [9], the stationary joint probability density $p(q_1, q_2, p_1, p_2)$ for the displacements and velocity of system (31) is obtained from $p(H_1, H_2)$ as follows:

$$p(q_1, q_2, p_1, p_2) = \frac{\bar{\omega}_1 \bar{\omega}_2}{4\pi^2} p(H_1, H_2)\Big|_{H_i = \frac{1}{2}p_i^2 + \frac{1}{2}\omega_i^2 q_i^2 + \frac{1}{4}\alpha_i q_i^4}. \tag{53}$$

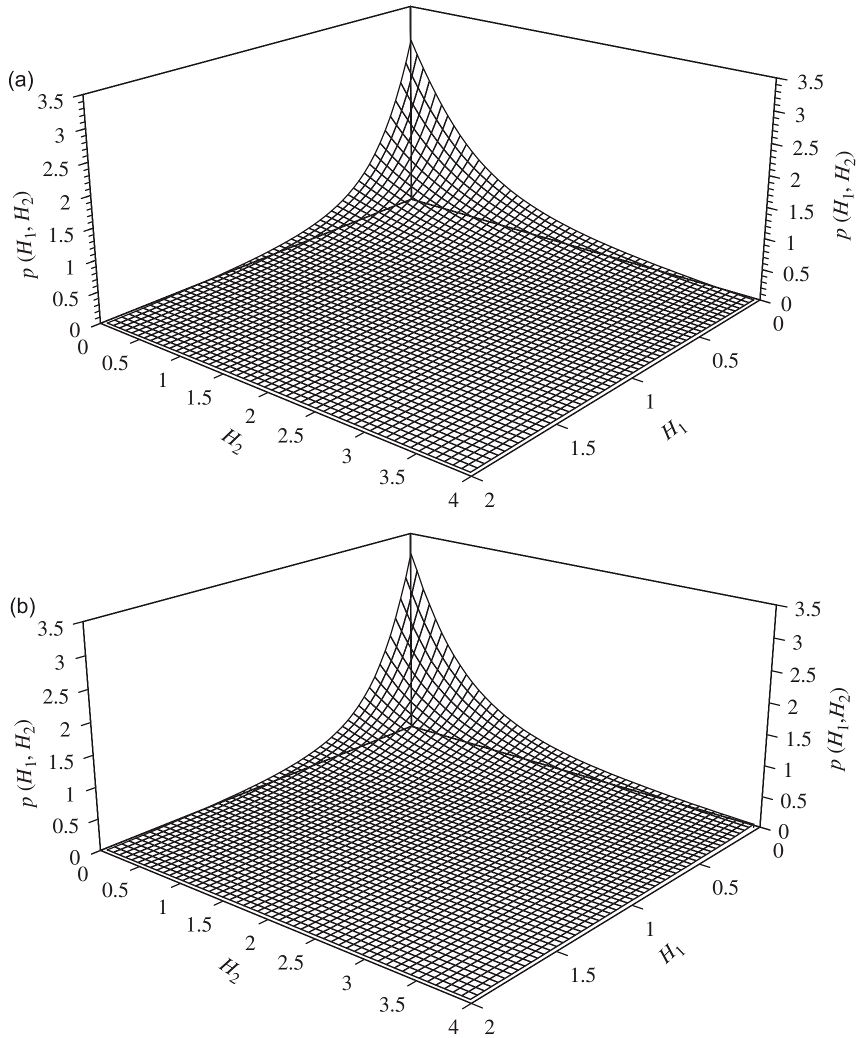

Fig. 2. Stationary probability density $p(H_1, H_2)$ of system (31). $\Omega_{11} = \Omega_{12} = \Omega_{21} = \Omega_{22} = 2$, $D_{11} = 0.1$, $D_{12} = 0.15$, $D_{21} = 0.05$ and $D_{22} = 0.15$. (a) From Eq. (48), (b) from digital simulation. The maximum absolute error is 0.24.

The other statistics of the stationary response of system (31) can then be obtained from Eq. (53). For example, the marginal stationary probability densities $p(H_1)$, $p(q_1)$, $p(q_1, p_2)$ and moments $E[H_1]$, $E[Q_1^2]$ can be obtained as follows:

$$p(H_1) = \int_0^\infty p(H_1, H_2)\,\mathrm{d}H_2, \quad E[H_1] = \int_0^\infty H_1 p(H_1)\,\mathrm{d}H_1,$$
$$p(q_1) = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty p(q_1, q_2, p_1, p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}q_2, \quad E[Q_1^2] = \int_{-\infty}^\infty q_1^2 p(q_1)\,\mathrm{d}q_1,$$
$$p(q_1, p_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty p(q_1, q_2, p_1, p_2)\,\mathrm{d}q_2\,\mathrm{d}p_1. \tag{54}$$

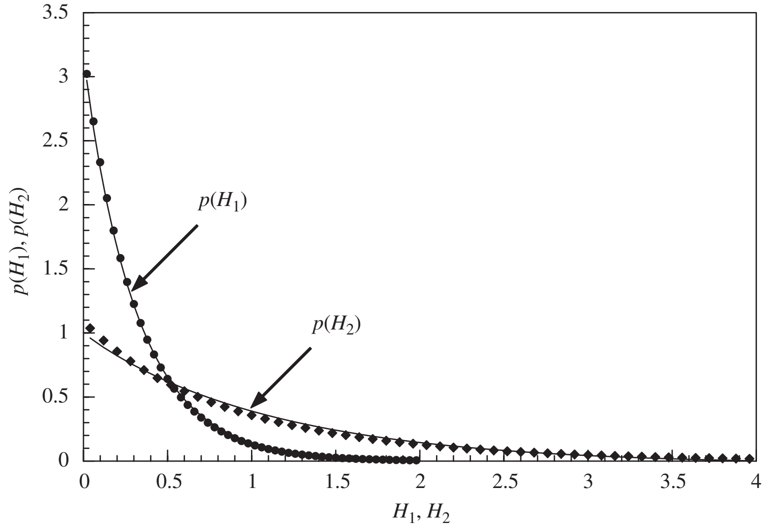

Fig. 3. Stationary marginal probability density $p(H_1)$ and $p(H_2)$ of system (31). The parameters are the same as those in Fig. 2. ——, analytical results; ●, ◆, results from simulation. The maximum absolute error is 0.053 for $p(H_1)$ and 0.076 for $p(H_2)$.

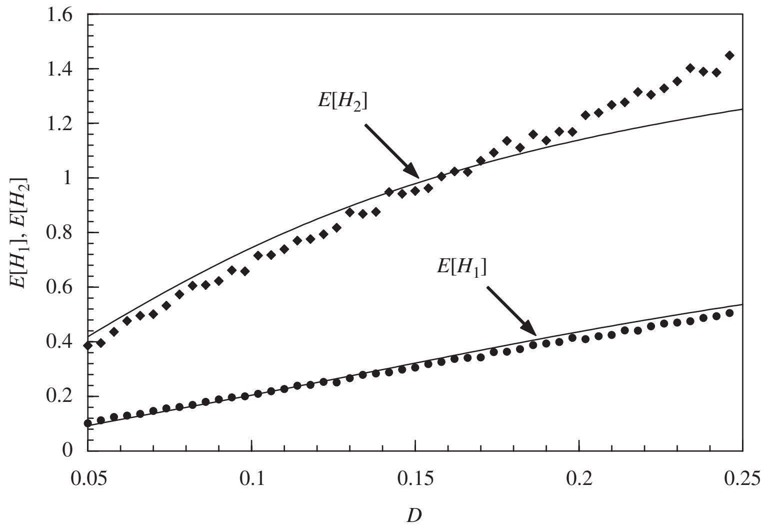

Fig. 4. Stationary mean value $E[H_1]$ and $E[H_2]$ of system (31) as function of excitation intensity $D_{12} = D_{22} = D$. The other parameters are the same as those in Fig. 2. ——, analytical results; ●, ◆, results from simulation. The maximum relatively error is 10.8% for $E[H_1]$ and 14.1% for $E[H_2]$.

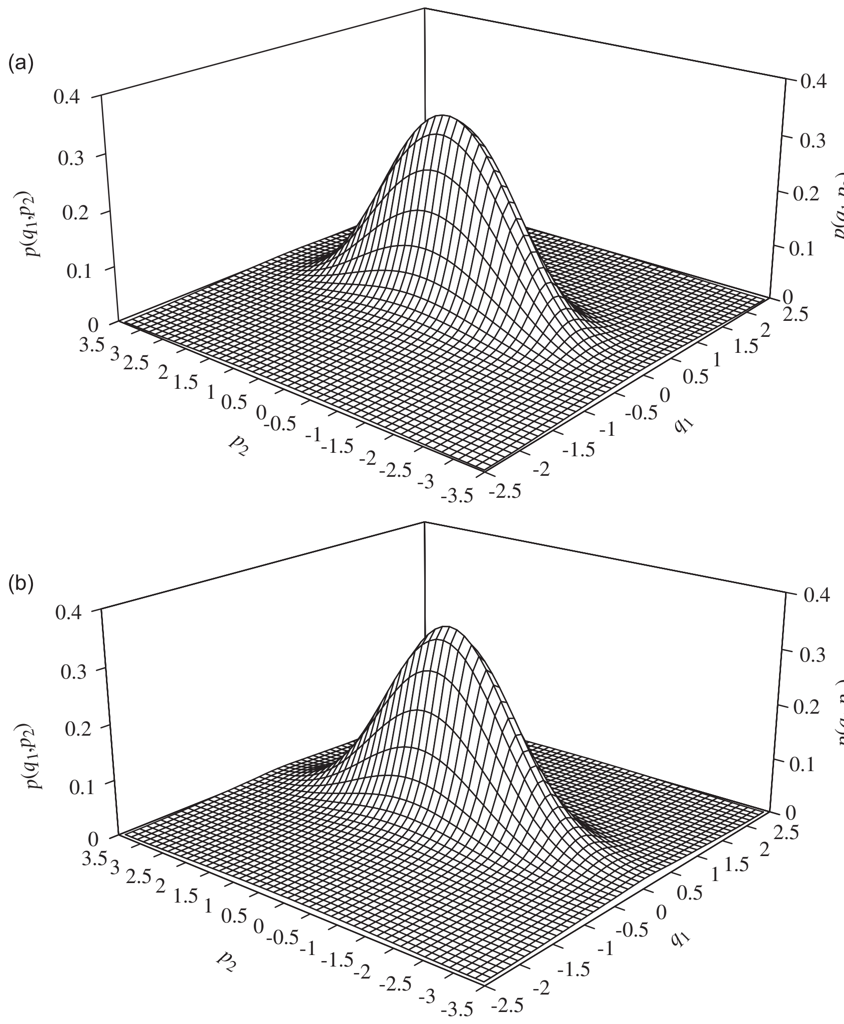

Fig. 5. Stationary marginal probability density $p(q_1, p_2)$ of system (31). The parameters are the same as those in Fig. 2. (a) From Eq. (54), (b) from simulation. The maximum absolute error is 0.02.

To check the accuracy of the results obtained by using the stochastic averaging method, Monte Carlo simulation of the original system (31) was performed. The sample functions of independent wide-band random processes $\xi_{ij}(t)$ were generated by inputting Gassian white noises to a linear filter. Then, the response of system (31) was obtained numerically by using the fourth-order Runge–Kutta method with time step 0.05. The long time solution after 10,000 steps was regarded as the stationary ergodic response and taken to perform the statistical analysis for obtaining the probability densities. Some numerical results obtained by using the proposed stochastic averaging method and from the digital simulation are shown in Figs. 2–7. The analytical results are denoted by the solid line —— and those from simulation are denoted by symbols ●, ◆. The system parameters used in all figures are $\beta_{11} = 0.2$, $\beta_{12} = 0.1$, $\beta_{21} = 0.1$, $\beta_{22} = 0.1$, $\omega_1 = 2.414$, $\omega_2 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 0.6$. It is seen from Figs. 2,3,5,6 that the analytical and simulation results are in rather good agreement. The errors are given in figure captions. Thus, the applicability of the proposed method in quasi-integrable Hamiltonian systems under wide-band random excitation is verified. It is noted, however, that since the stochastic averaging method is an approximate technique and it is established under some assumptions, the proposed method yield only approximate results and is applicable only for certain parameter domain which depends upon the error allowed.

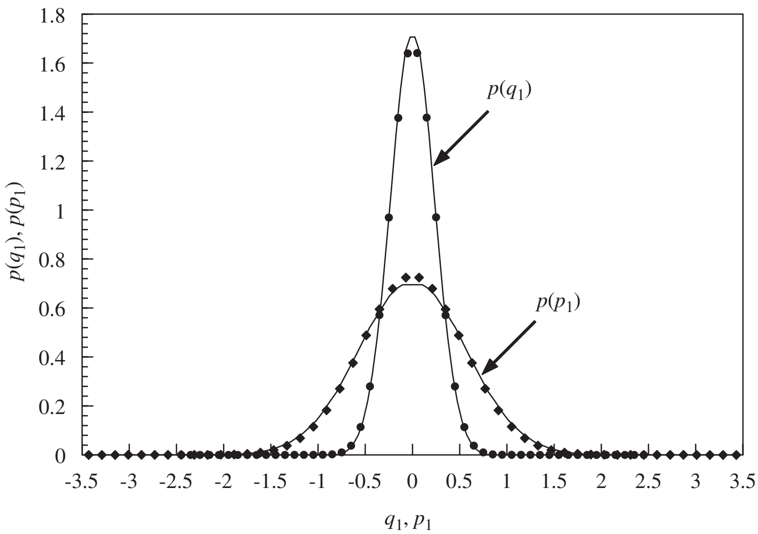

Fig. 6. Stationary marginal probability densities $p(q_1)$ and $p(p_2)$ of system (31). The parameters are the same as those in Fig. 2. ——, analytical results; ●, ◆, results from simulation. The maximum absolute error is 0.029 for $p(p_1)$ and 0.037 for $p(q_1)$.

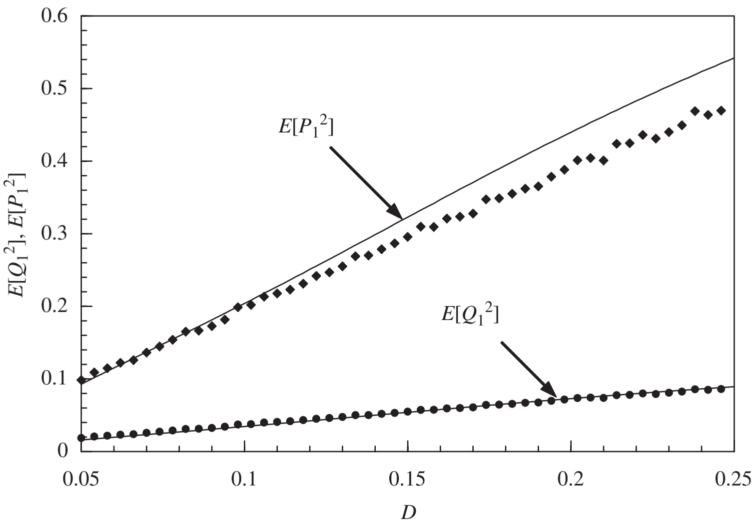

Fig. 7. Stationary mean-square value $E[Q_1^2]$ and $E[P_1^2]$ of system (31) as function of excitation intensity $D_{12} = D_{22} = D$. The other parameters are the same as those in Fig. 2. ——, analytical results; ●, ◆, results from simulation. The maximum relatively error is 15.7% for $E[Q_1^2]$ and 15.1% for $E[P_1^2]$.

4. Concluding remarks

In the present paper, a stochastic averaging method for quasi-integrable Hamiltonian systems subject to external and/or parametric wide-band random excitations has been proposed. Both the averaged Itô stochastic differential equations governing amplitudes of displacements and those governing the energies of various degrees of freedom and their associated FPK equations have been established. By numerically solving the FPK equation, all responses statistics can be obtained. An example has been worked out in detail to illustrate the application of the proposed procedure. The comparison between the analytical results and those from digital simulation has showed that the proposed method works quite well.

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